Differential Geometry III: Riemannian Geometry

8.1 Let (X, d_1) and (Y, d_2) be two metric spaces and let $f : X \to Y$ be a continuous function. Assume that the following three conditions are satisfied:

1. f is an *open map*, i.e. $f(\mathcal{U})$ is an open subset of Y for any open set $\mathcal{U} \subset X$.
2. f is a *proper map*, i.e. for any $\mathcal{K} \subset Y$ which is compact, $f^{-1}(\mathcal{K})$ is a compact subset of X.
3. Y is connected.

Show that f is surjective.

8.2 Let $(\mathcal{M}, g)$ be a *homogeneous* Riemannian manifold (i.e. for any two points $p, q \in \mathcal{M}$, there exists an isometry $F : \mathcal{M} \to \mathcal{M}$ such that $F(p) = q$). Show that $(\mathcal{M}, g)$ is complete.

Hint: You might want to use the fact that isometries map geodesics to geodesics, to infer that every point on $\mathcal{M}$ has the same injectivity radius. Under this condition, can a maximal geodesic be incomplete?

8.3 A Riemannian manifold $(\mathcal{M}, g)$ is called *isotropic* if, for every $p \in \mathcal{M}$ and every $v_1, v_2 \in T_p\mathcal{M}$ with $\|v_1\| = \|v_2\|$, there exists an isometry F of $(\mathcal{M}, g)$ with $F(p) = p$ and $dF(v_1) = v_2$ (in other words, around p the space $(\mathcal{M}, g)$ "looks the same" in every direction).

(a) Can you find an example of an isotropic Riemannian manifold? An example of a homogeneous but not isotropic Riemannian manifold?

(b) Show that a connected, complete and isotropic Riemannian manifold $(\mathcal{M}, g)$ is also homogeneous (see Ex. 8.2). *Hint: For any $p, q \in \mathcal{M}$, let x be the midpoint of a geodesic segment connecting p to q (why does it exist?); consider the set of isometries that fix x.*

(c) Show that the assumption on $(\mathcal{M}, g)$ being complete above is redundant, i.e. that every connected and isotropic Riemannian manifold is also complete (and, hence, homogeneous). *Hint: If $\gamma : (a, b) \to \mathcal{M}$, $0 \in (a, b)$, is a maximal geodesic, show that the isotropic condition implies that $a = -b$. Note that this should be true for all points along the geodesic.*

8.4 Let $F : (\mathcal{M}, g) \to (\mathcal{N}, h)$ be a local isometry between two Riemannian manifolds (recall that a local isometry is a map for which $dF|_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$ is 1-1 and onto for all $p \in \mathcal{M}$ and $F^*h = g$). Assume that $\mathcal{N}$ is connected and $(\mathcal{M}, g)$ is complete. Show that F is onto and that $(\mathcal{N}, h)$ is also complete. Is F necessarily $1 - 1$?